Mean-variance hedging and optimal investment in Heston's model with correlation

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Outline of the talk

- New general results for mean-variance hedging problem
  - Černý and Kallsen (2007b) On the Structure of General M-V Hedging Strategies, *Annals of Probability*
- New characterization of variance-optimal measure
  - Černý and Kallsen (2008a) A Counterexample Concerning VOMM, *Mathematical Finance*, 18(2)
- Relation to globally mean-variance efficient portfolios
- Relation to good-deal pricing
- Application to the Heston model with leverage
  - Černý and Kallsen (2008b) *Mathematical Finance*, 18(3)

Basic problem

- Finite time horizon T
- Initial endowment c
- "Admissible" trading strategy $\vartheta \in \overline{\Theta}$
- Contingent claim $H \in L^2(P)$
- Discounted stock price S
- Value of self-financing strategy $\vartheta \bullet S_T := \int_0^T \vartheta_t dS_t$

$$\inf_{\vartheta \in \overline{\Theta}} E\left((c + \vartheta \bullet S_T - H)^2\right)$$

- Optimal strategy denoted by $\varphi(c, H)$

Key elements of the new approach

- Best of many worlds
  - General semimartingale model
  - Minimal assumptions
  - Explicit formulae for all quantities of interest
  - Simple interpretation of results (no numeraire change)
- We introduce the opportunity process

$$L_t := \inf_{\vartheta} E\left(\left(1 - 1_{]\!]t,T]\!]}\vartheta \bullet S\right)^2 \middle| \mathscr{F}_t \right),$$

- $\sqrt{1/L_t - 1}$ is the maximal Sharpe ratio attainable by dynamic trading in the time interval $(t, T]$

Computation of optimal hedge I

1. Compute $K := L_-^{-1} \bullet L$ and its canonical decomposition $K = B^K + M^K$ under P
2. Define the opportunity-neutral measure $P^\star$ by setting
$$\frac{dP^\star}{dP} := \mathscr{E}\left(\frac{1}{1+\Delta B^K} \bullet M^K\right)$$
3. Evaluate the $P^\star$ canonical decomposition of S
$$S = S_0 + B^{S\star} + M^{S\star}$$
4. Define
$$\tilde{a} := dB^{S\star}/d\langle S, S\rangle^{P^\star}, \qquad \hat{a} := dB^{S\star}/d\langle M^{S\star}, M^{S\star}\rangle^{P^\star}$$

Computation of optimal hedge II

5. Compute the variance-optimal measure $Q^\star$ as the minimal martingale measure relative to $P^\star$

$$\frac{dQ^\star}{dP^\star} = \mathscr{E}\left(-\hat{a} \bullet M^{S\star}\right)$$

6. Define mean value process V as a conditional expectation under $Q^\star$
7. Obtain ξ from the F–S decomposition of H under $P^\star$

$$\xi_t = \frac{d\langle V, S\rangle_t^{P^\star}}{d\langle S, S\rangle_t^{P^\star}} = \frac{d\langle M^{V\star}, M^{S\star}\rangle_t^{P^\star}}{d\langle M^{S\star}, M^{S\star}\rangle_t^{P^\star}}$$

8. Optimal hedging strategy $\varphi(c, H)$ is given by

$$\varphi(c, H) = \xi + \tilde{a}(V_- - c - \varphi(c, H) \bullet S_-)$$

Opportunity-neutral measure $P^\star$

- $P^\star$ is not a martingale measure, but it is equivalent to P
- Under $P^\star$ one effectively returns back to the deterministic opportunity set discussed in Schweizer (1994)
- Intuitively, $P^\star$ assigns lower conditional probability to the states with higher Sharpe ratio, in direct proportion to the value of L in different states

$$
\begin{aligned}
A &\in \mathscr{F}_{t+1} \\
P^\star(A)|\mathscr{F}_t &= \frac{E(1_A L_{t+1}|\mathscr{F}_t)}{E(L_{t+1}|\mathscr{F}_t)}
\end{aligned}
$$

- More on discrete-time hedging in Černý and Kallsen (2007a) "Hedging by sequential regressions revisited", *to appear in MF*

Alternative representations of $Q^\star$

- Hipp (1993), Schweizer (1996)

$$\frac{dQ^\star}{dP} = \frac{\mathscr{E}\left(-\tilde{a} \bullet S\right)_T}{E\left(\mathscr{E}\left(-\tilde{a} \bullet S\right)_T\right)}$$

- New result: the density process of $dQ^\star / dP$ equals

$$\mathscr{E}(K)\mathscr{E}(-\tilde{a} \bullet S)$$

- Černý and Kallsen (2008a) counterexample paper has more details with links to the representation equation of Biagini et al. (2000) and Hobson (2004)

Computation of L and $\tilde{a}$

- A candidate solution for L is characterized explicitly from the drift condition

$$B^K = \hat{a} \bullet B^{S\star}$$

- Economic meaning: expected rate of increase in the opportunity process = square of instantaneous Sharpe ratio under the opportunity-neutral measure
- This immediately yields PDEs reported, for example, in Laurent and Pham (1999), Biagini et al. (2000) and Hobson (2004)

Necessary and sufficient conditions

- Sufficient conditions for a candidate solution to be a true solution include
  1. $\mathscr{E}(-a \bullet S) \in \mathscr{S}^2$ – Schweizer's set
  2. $\sup_{t\in[0,T]}\{\mathscr{E}(-a \bullet S)_t\}$ is in $L^2(P)$,
  3. $\sup_\sigma\{E\left(\mathscr{E}(-(a\mathbf{1}_{]\!]\tau,T]\!]}) \bullet S)^2_\sigma\right)\} < \infty$.
- Necessary conditions state that $L\mathscr{E}(-a \bullet S)$ and $L(\mathscr{E}(-a \bullet S))^2$ must be true martingales. These are not sufficient in general, cf. Černý and Kallsen (2008a) counterexample paper

Hedging error of optimal strategy

- The sq. hedging error of optimal strategy for fixed initial endowment c equals

$$E\big((c + \varphi \bullet S_T - H)^2\big) = L_0(c - V_0)^2 + \varepsilon_0^2$$

- ε_0^2 is the minimal unconditional expected sq. hedging error of a perfectly balanced initial position

$$\varepsilon_0^2 = E(L \bullet \langle V - \xi \bullet S, V - \xi \bullet S\rangle_T^{P^\star})$$

Globally M-V efficient portfolios

- Link to globally mean-variance efficient portfolios
  - Easy identity

$$\min_{\alpha \in \mathbb{R}} E\left((1-\alpha X)^2\right) = 1 - \frac{\mu_X^2}{E(X^2)} = \frac{1}{1+\mathrm{SR}^2(X)}$$

  - Consequence

$$\min_{\vartheta \in \overline{\Theta}} \frac{1}{1+\mathrm{SR}^2(\vartheta \bullet S_T)} = \min_{\vartheta \in \overline{\Theta}} E\left((1-\vartheta \bullet S_T)^2\right)$$

- Take $H=1$, initial wealth $=0$ then φ is globally mean-variance efficient with moments

$$\begin{aligned} E(\varphi(0,1)\bullet S_T) &= 1-L_0, \\ \mathrm{Var}(\varphi(0,1)\bullet S_T) &= L_0(1-L_0) \end{aligned}$$

Good-deal price bounds

- Recall the maximal Sharpe ratio from trading in the stock equals $\mathrm{SR}_S^2 = 1/L_0 - 1$
- Suppose one can buy or sell the contingent claim H at price C_0
- Then it is optimal to sell η units of the contingent claim

$$\eta = \frac{C_0 - V_0}{\varepsilon_0^2} \frac{1}{1 + \mathrm{SR}_{S,H}^2}$$

- The resulting Sharpe ratio is

$$\mathrm{SR}_{S,H}^2 = 1/L_0 - 1 + (C_0 - V_0)^2/\varepsilon_0^2.$$

- Precursor to these results is in Duffie and Richardson (1991) and Schweizer (1996)

Formulation

- Change in stock price and volatility are correlated

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu Y_t^2 dt + Y_t dW_t,\\ dY_t^2 &= \left(\zeta_0 + \zeta_1 Y_t^2\right) dt + \sigma Y_t \left(\varrho dW_t + \sqrt{1-\varrho^2} dU_t\right).\end{aligned}$$

- Instantaneous Sharpe ratio equals μY
- Y is an autonomous diffusion
- Opportunity set is a deterministic function of Y
- Conditions on ζ_0 and ζ_1 ensure steady state distribution under of Y under P (cf. Cox et al. (1985)).

$$\zeta_0 \geq \sigma^2/2, \quad \zeta_1 < 0$$

Differential characteristics

- Introduce differential characteristics b, c

$$dB^S = b^S dt, \quad d[S,S] = c^S dt, \quad d[S,Y^2] = c^{SY^2} dt, \text{ etc.}$$

- b, c interpreted as drift and instantaneous variance-covariance matrix

$$\begin{pmatrix} b^{Y^2} \\ b^S \end{pmatrix} = \begin{pmatrix} \zeta_0 + \zeta_1 Y^2 \\ \mu S Y^2 \end{pmatrix},$$

$$\begin{pmatrix} c^{Y^2} & c^{Y^2 S} \\ c^{SY^2} & c^S \end{pmatrix} = \begin{pmatrix} \sigma^2 Y^2 & \varrho\sigma S Y^2 \\ \varrho\sigma S Y^2 & S^2 Y^2 \end{pmatrix}.$$

Computation I

- We guess a candidate opportunity process in the form

$$L = \exp(\varkappa_0 + \varkappa_1 Y^2),$$

where $\varkappa_0$ and $\varkappa_1$ are deterministic functions

- Evaluate the modified mean-variance trade-off $dK := dL/L$ using the Itô formula

$$dK = (\varkappa_0' + Y^2\varkappa_1' + \frac{1}{2}\sigma^2 Y^2 \varkappa_1^2)dt + \varkappa_1 dY^2$$

- Write down the optimality condition

$$b^K = (b^S + c^{KS})^2/c^S$$

Computation II

- Collect different powers of Y

$$
\begin{aligned}
-\varkappa_0' &= \zeta_0 \varkappa_1, \\
-\varkappa_1' &= -\mu^2 + (\zeta_1 - 2\varrho\sigma\mu)\,\varkappa_1 + \frac{1}{2}\sigma^2\left(1 - 2\varrho^2\right)\varkappa_1^2,
\end{aligned}
$$

- Boundary condition from $L_T = 1$: $\varkappa_1(T) = \varkappa_0(T) = 0$
- Solve for $\varkappa_1, \varkappa_0$
- Solutions are explicit but $\varkappa_1(0)$ may explode to $-\infty$ for finite $T^\star$
- Interpretation:

$$
L_0 \to 0 \Rightarrow \text{ unconditional Sharpe ratio } \to \infty
$$

Verification I

- We now have a candidate for L and the corresponding candidate for $a := \hat{a} = \tilde{a}$

$$a = (b^S + c^{KS})/c^S = (\mu + \varkappa_1 \sigma \varrho)/S$$

- Define the candidate VOMM $Q^\star$

$$\frac{dQ^\star}{dP} = \mathscr{E}(-a \bullet S_T)/L_0$$

- Need to check that the local martingale $L\mathscr{E}(-a \bullet S)$ is a true martingale so that $Q^\star$ has mass 1
- Check that the local martingale $L\mathscr{E}(-a \bullet S)^2$ is a true martingale so that $Q^\star$ is a martingale measure with square-integrable density

Verification II

- This does not imply that $Q^\star$ is the true VOMM! (cf. Černý and Kallsen 2008a counterexample paper)
- One has to show $\mathscr{E}(-a \bullet S)$ is a true martingale for all EMM with square-integrable density
- Strategy
  1. Write down the Novikov condition under a general EMM Q
  2. Apply the Hölder inequality to obtain P-expectation
  3. Estimate the P-expectation using the theory of affine processes (Duffie et al. 2003)
- We can rigorously prove optimality for T small enough, but not for all $T < T^\star$

$P^\star$ dynamics of price and volatility

$$
\begin{aligned}
\frac{dP^\star}{dP} &= \mathscr{E}(M^K) \\
b^{S\star} &= b^S + c^{SK} = (\mu + \varkappa_1 \sigma \varrho)\, Y^2 \\
b^{Y^2\star} &= b^{Y^2} + c^{Y^2 K} = \zeta_0 + \zeta_1^\star Y^2 \\
\zeta_1^\star &:= \zeta_1 + \sigma^2 \varkappa_1
\end{aligned}
$$

- With $\varrho < 0$ the mean return on the stock is higher
- Volatility exhibits stronger mean reversion and the steady state mean is lower
- Dynamically optimal investment a exceeds the myopic investment $\lambda = b^S / c^S$

$Q^\star$ dynamics of price and volatility

$$
\begin{aligned}
\frac{dQ^\star}{dP^\star} &= \mathscr{E}(-a \bullet M^{S\star}) \\
b^S_{Q^\star} &= b^{S\star} - ac^S = 0 \\
b^{Y^2}_{Q^\star} &= b^{Y^2\star} - ac^{Y^2 S} = \zeta_0 + \hat{\zeta}_1^\star Y^2 \\
\hat{\zeta}_1^\star &:= \zeta_1^\star - \varrho\sigma(\mu + \kappa_1 \varrho\sigma)
\end{aligned}
$$

- Volatility mean reversion under $Q^\star$ is lower compared to $P^\star$
- Comparison of coefficients under $Q^\star$ and P is ambiguous.

Mean value process

- Denote $f_i := \partial f / \partial x_i, f_{ij} := \partial^2 f / (\partial x_i \partial x_j)$
- Define

$$V_t := E^{Q^\star}(H|\mathcal{F}_t).$$

Proposition (Heath and Schweizer 2000)

If the contingent claim H is given by $g(Y_T^2, S_T)$ where g is a bounded continuous function then $V_t = f(T-t, Y_t^2, S_t)$ for $f \in \mathcal{C}^{1,2,2}$ and f is the unique classical solution of the PDE

$$0 = -f_1 + \left(\zeta_0 + \hat{\zeta}_1^\star y\right) f_2 + \frac{1}{2} y \left(\sigma^2 f_{22} + 2\varrho\sigma s f_{23} + s^2 f_{33}\right)$$

with the boundary condition $f(0, y, s) = g(y, s)$.

Hedging strategy

- Pure hedging coefficient satisfies

$$\xi_t = \frac{c_t^{VS}}{c_t^S} = f_3(T - t, Y_t^2, S_t) + \varrho\sigma f_2(T - t, Y_t^2, S_t)/S_t.$$

- Pure hedge has two components:
  1. the standard delta hedge using the representative agent price V_t,
  2. leverage component exploiting the correlation of the representative agent price with the volatility process

Hedging error

- Instantaneous squared error γ

$$\gamma_t := c_t^V - \frac{\left(c_t^{SV}\right)^2}{c_t^S} = \left(f_2(T-t, Y_t^2, S_t)\right)^2 \sigma^2 Y_t^2 \left(1-\varrho^2\right).$$

- The minimal squared hedging error with initial capital V_0 satisfies

$$\begin{aligned}
\varepsilon_0^2 &:= E\left(\left(V_0 + \varphi(V_0, H) \bullet S_T - H\right)^2\right) \\
&= E\left(\int_0^T L_t \gamma_t dt\right) \\
&= \sigma^2\left(1-\varrho^2\right) \times \\
&\quad E\left(\int_0^T e^{\varkappa_0(t)+\varkappa_1(t)Y_t^2} Y_t^2 \left(f_2(T-t, Y_t^2, S_t)\right)^2 dt\right).
\end{aligned}$$

Final remarks

- $V_0, \xi_0, \varepsilon_0$ have a Fourier transform representation (given in the paper)
- Testable predictions
  - Optimal dynamic M-V portfolio allocation
  - Price deviations from V_0 (global good-deal bounds)

Final remarks

- All papers available from SSRN
- These slides can be found at www.martingales.info

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