On the structure of general mean–variance hedging strategies

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Basic problem

- Finite time horizon T
- Initial endowment c
- "Admissible" trading strategy ϑ
- Contingent claim $H \in L^2(P)$
- Discounted stock price S

$$\inf_{\vartheta} E\left((c + \vartheta \bullet S_T - H)^2\right)$$

Martingale case

Föllmer and Sondermann (1986) solve this problem using Galtchouk-Kunita-Watanabe decomposition

$$\begin{aligned} V &= V_0 + \varphi \bullet S + R, \\ V_T &= H, \\ R & \text{ is a } P\text{-martingale strongly orthogonal to } S, \end{aligned}$$

The GKW decomposition can be computed explicitly

$$\begin{aligned} V_t &= E\left(H|\mathscr{F}_t\right) \\ \varphi_t &= \frac{d\left\langle V,S\right\rangle_t^P}{d\left\langle S\right\rangle_t^P} \end{aligned}$$

V is called the mean value process

Non-martingale case

Definition (Jacod 1979, Back 1991)

Let $\langle X, Y\rangle^P$ be the predictable compensator of $[X, Y]$ under measure P if X and Y are semimartingales such that $[X, Y]$ is special.

Assume $S = S_0 + M^S + A^S$ under P

Definition (Schweizer 1992, 1994)

- Myopic mean–variance portfolio process $\hat{\lambda} := dA^S_t / d\left\langle M^S\right\rangle^P_t$
- Mean–variance tradeoff $\hat{K} := \hat{\lambda} \bullet A^S$
- Minimal martingale measure

$$dQ/dP := \mathscr{E}\left(-\hat{\lambda} \bullet M^S\right) = \mathscr{E}\left(\hat{K} - \hat{\lambda} \bullet S\right), \qquad (1.1)$$

Relationship between $\hat{K}$ and $\tilde{K}$

Definition (Schweizer 1994)

- Myopic utility portfolio process

$$\tilde{\lambda} := dA^S_t / d\left\langle S \right\rangle^P_t$$

- Quadratic utility tradeoff

$$\tilde{K} := \tilde{\lambda} \bullet A^S$$

- $\hat{\lambda}$ and $\tilde{\lambda}$ coincide if and only if S is quasi left-continuous
- One can pass from $\hat{\lambda}$ to $\tilde{\lambda}$ and vice versa

$$\tilde{\lambda} = \frac{\hat{\lambda}}{1 + \Delta \hat{K}}$$

$$\mathscr{E}(\hat{K})\mathscr{E}(-\tilde{K}) = 1$$

Theory

Theorem (Schweizer 1994)

When $\tilde{K}$ is deterministic we have

$$\varphi = \xi + \tilde{\lambda}\left(V_{-} - c - \varphi \bullet S_{-}\right),$$

where ξ is the integrand of S in the Föllmer-Schweizer decomposition of H

$$\begin{aligned} V &= V_0 + \xi \bullet S + R, \\ V_T &= H, \\ R & \text{ is a } P\text{-martingale strongly orthogonal to } M^S, \end{aligned}$$

If the minimal martingale measure Q is equivalent to P then

$$V_t = E^Q\left(H \middle| \mathscr{F}_t\right)$$

Computation

- In general not obvious how to evaluate the F–S decomposition
  - Colwell and Elliott (1993) – jump diffusion
  - Hubalek et al. (2004) – exponential Lévy model
- When S is continuous Föllmer and Schweizer (1991) show this boils down to GKW decomposition under minimal measure Q

$$\xi = d\left\langle V, S\right\rangle_t^Q / d\left\langle S\right\rangle_t^Q .$$

- We make 2 contributions in this area
  - We redefine the conditional expectation under Q so that $V_t = E^Q\left(H|\mathscr{F}_t\right)$ works for signed Q, cf. Choulli et al. (1998)
  - We give an explicit general formula for ξ

$$\xi_t = \frac{d\left\langle V, S\right\rangle_t^P}{d\left\langle S\right\rangle_t^P} = \frac{d\left\langle M^V, M^S\right\rangle_t^P}{d\left\langle M^S\right\rangle_t^P}$$

Theory

- Schweizer (1996) shows the structure of the solution remains similar when $\hat{K}$ is stochastic
- Specifically, there is so called adjustment process $\tilde{a}$ such that

$$\varphi = \xi + \tilde{a}\,(V_- - c - \varphi \bullet S_-)\,.$$

- In general there are no immediate recipes for computation of $\tilde{a}$, V, ξ

Theory – Continuous processes

Suppose that $\tilde{a}$ is already given. Rheinländer and Schweizer (1997) give three simple steps to find V, ξ

1. Define the variance-optimal measure $Q^\star$

$$\frac{dQ^\star}{dP} = \frac{\mathscr{E}\left(-\tilde{a} \bullet S\right)_T}{E\left(\mathscr{E}\left(-\tilde{a} \bullet S\right)_T\right)}$$

2. Compute the $Q^\star$-martingale V

$$V_t = E^{Q^\star}\left(H | \mathscr{F}_t\right)$$

3. Evaluate the GKW decomposition of H under $Q^\star$

$$\xi_t = \frac{d\left\langle V, S\right\rangle_t^{Q^\star}}{d\left\langle S\right\rangle_t^{Q^\star}}$$

Theory – Numeraire change

- Suppose that $\tilde{a}$ is already given.
- For continuous S Gourieroux et al. (1998) transform the original problem so that ξ can be evaluated via GKW decomposition of the payoff expressed in new units
- Arai (2005) shows this will (remarkably !) work for general semimartingales as long as $Q^\star$ is an equivalent measure
- In practice this approach requires a fair amount of extra computations

Computation – Adjustment process $\tilde{a}$

- Laurent and Pham (1999) use dynamic programming in diffusion stochastic volatility models without leverage
- Biagini et al. (2000) formulate representation equation involving $\hat{K}$
- Hobson (2004) uses an equivalent expression to tackle Heston (1993) with leverage

$$\hat{K}_T = \eta \bullet \left(M^S + 2A^S\right) - \frac{1}{2}\eta^2 \bullet \left\langle M^S \right\rangle + \zeta \bullet R + \frac{1}{2}\zeta^2 \bullet \langle R \rangle + c$$

- Lim (2004) uses jump diffusion setting where asset price characteristics are adapted to a Brownian filtration

Choice of trading strategies

- The existence of the portfolio weights $\tilde{a}$ and of the optimal hedging strategy φ depends crucially on the chosen class of "admissible" trading strategies Θ.
- If the linear subspace of wealth distributions $\vartheta \bullet S$ generated by $\vartheta \in \Theta$ is closed in $L^2(P)$ the optimal hedging strategy exists for any contingent claim $H \in L^2(P)$.
- In the original approach due to Schweizer (1994) the marketed subspace may not be closed
- Sufficient conditions for closedness are examined in Monat and Stricker (1995), Delbaen et al. (1997) and Choulli et al. (1998). They are used in Arai (2005).

Choice of trading strategies

- In contrast, Delbaen and Schachermayer (1996a) design admissible strategies in such a way that the marketed subspace is closed in L^2 by default.
- For continuous processes the marketed subspace of DS96a is exactly the L^2 closure of Schweizer's marketed subspace. This framework is used in Gourieroux et al. (1998).
- For discontinuous processes the framework of DS96a is less pleasant
- In this case Θ is "too wide" because signed measures are excluded from the duality between self-financing strategies and martingale measures.
- We fix the definition of admissible strategies.

Martingale measures for mean–variance preferences

- It is well known that the CAPM pricing kernel is linear in market returns and it may in principle become negative.
- This phenomenon arises already in one-period finite state models
- It is intimately related to the existence of a bliss point for quadratic utility function, cf. Černý (2004, Chapter 3)
- To build a good mean–variance hedging theory one must therefore consider signed martingale measures.

Conditional expectation under a signed measure

- In a dynamic context signed measures create additional conceptual problems
- Let Q be a signed measure, $Z_T := dQ/dP$, with density process $Z_t := E(Z_T|\mathscr{F}_t)$
- Since the density may become 0 with positive probability, one cannot write

$$E^Q(X|\mathscr{F}_t) := E(XZ_T|\mathscr{F}_t)/Z_t$$

- The solution is to restart Z at 1 every time it jumps to 0.

Conditional expectation under a signed measure I

- If Q has a log density process N then $\mathscr{E}(N) = Z$ and one can define

$$E^Q(X|\mathscr{F}_t) := E(X\mathscr{E}(N_T - N_t)|\mathscr{F}_t)$$

- The above definition appeared in Černý (2004a) for $|\Omega| < \infty$ in the context of VOMM
- It is closely related to the notion of $\mathscr{E}$-martingales introduced in Choulli et al. (1998)

Conditional expectation under a signed measure II

- When Z is positive N and Z are one-to-one
- When Z is signed N determines Z but not vice versa
- Not all signed martingale measures have a log density
- Extending arguments of DS96b we show that VOMM $Q^{\star}$ can be associated with a log density such that

$$V_t = E^{Q^{\star}}(H|\mathscr{F}_t)$$

Notation and assumptions

- Filtered probability space $\left(\Omega, F, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P\right)$
- Fixed time horizon $T \in \mathbb{R}_+$
- d securities $S = (S_t^1, \ldots, S_t^d)_{t\in[0,T]}$
- Two assumptions based on DS96a

Assumption

1. *S is in $L^2(P)$, that is*

$$\sup\left\{E\left(\left(S_\tau^i\right)^2\right) : \tau \text{ stopping time, } i = 1, \ldots, d\right\} < \infty.$$

2. *There is an equivalent martingale measure $\bar{Q}$ with square integrable density, i.e.*

$$\bar{Q} \sim P, \quad E\left(d\bar{Q}/dP\right)^2 < \infty, \quad S \text{ is a } \bar{Q}\text{-martingale.}$$

Key elements of the new approach

- Best of three worlds
  - General semimartingale model
  - Explicit formulae for all quantities of interest
  - Simple interpretation of results (no numeraire change)
- We define the opportunity process

$$L_t := \inf_{\vartheta} E\left(\left(1 - 1_{]\!]t,T]\!]}\vartheta \bullet S\right)^2 \middle| \mathscr{F}_t\right),$$

- We show that $\sqrt{1/L_t - 1}$ is the maximal Sharpe ratio attainable by dynamic trading in the time interval $(t, T]$
- Let K be the stochastic logarithm of L with the Doob–Meyer decomposition $K = A^K + M^K$ under P
- We define the opportunity-neutral measure $P^\star$ by setting

$$\frac{dP^\star}{dP} := \mathscr{E}\left(\frac{1}{1 + \triangle A^K} \bullet M^K\right)$$

Opportunity-neutral measure $P^\star$

- $P^\star$ is not a martingale measure, but it is equivalent to P
- Under $P^\star$ one effectively returns back to the deterministic opportunity set discussed in Schweizer (1994)
- Intuitively, $P^\star$ assigns lower conditional probability to the states with higher Sharpe ratio, in direct proportion to the value of L in different states

$$
\begin{aligned}
A &\in \mathscr{F}_{t+1} \\
P^\star(A)|\mathscr{F}_t &= \frac{E(1_A L_{t+1}|\mathscr{F}_t)}{E(L_{t+1}|\mathscr{F}_t)}
\end{aligned}
$$

Opportunity-neutral measure $P^\star$

We prove that S is $P^\star$-special. Let $S = S_0 + M^{S\star} + A^{S\star}$ be the canonical decomposition under $P^\star$

- $\hat{a}$ and $\tilde{a}$ relate to $P^\star$ the same way as $\hat{\lambda}$ and $\tilde{\lambda}$ relate to P

$$\hat{a}_t = dA_t^{S\star} / d\left\langle M^{S\star} \right\rangle_t^{P^\star} \qquad \tilde{a}_t = dA_t^{S\star} / d\left\langle S \right\rangle_t^{P^\star}$$

- Variance-optimal measure is the $P^\star$ minimal martingale measure

$$\frac{dQ^\star}{dP^\star} = \mathscr{E}\left(-\hat{a} \bullet M^{S\star}\right)$$

- ξ is obtained from the F–S decomposition of H under $P^\star$

Alternative representations of $Q^\star$

- Hipp (1993), Schweizer (1996)

$$\frac{dQ^\star}{dP} = \frac{\mathscr{E}\left(-\tilde{a} \bullet S\right)}{E\left(\mathscr{E}\left(-\tilde{a} \bullet S\right)\right)}$$

- New result

$$E\left(\left.\frac{dQ^\star}{dP}\right| \mathscr{F}_t\right) = \mathscr{E}(K)_t \mathscr{E}\left(-\tilde{a} \bullet S\right)_t$$

- Representation equation Biagini et al. (2000), Hobson (2004)

$$\mathscr{E}(K)\mathscr{E}\left(-\tilde{a} \bullet S\right) = \mathscr{E}\left(-\hat{\lambda} \bullet M^S + \zeta \bullet R\right)$$

Computation of L and $\tilde{a}$

- Once we have obtained L the adjustment process $\tilde{a}$ is computed without extra effort ($L \to K \to P^\star \to \tilde{a}$)
- A candidate solution for L is characterized explicitly from the drift condition

$$A^K = \hat{a} \bullet A^{S\star} = \hat{K}^\star$$

- This immediately yields PDEs reported, for example, in Laurent and Pham (1999), Biagini et al. (2000) and Hobson (2004)
- Sufficient condition for a candidate solution to be a true solution is $\mathscr{E}(-\tilde{a} \bullet S)$ in $L^2(P)$
- Necessary conditions state that $L\mathscr{E}(-\tilde{a} \bullet S)$ and $L(\mathscr{E}(-\tilde{a} \bullet S))^2$ must be true martingales.

Hedging error of optimal strategy

- The "price process" V may not be locally square integrable under P but it is always locally square integrable under $P^\star$
- The crucial step is to show LV^2 is a submartingale
- The optimal hedging error equals

$$
\begin{aligned}
& E\left((v_0 + \varphi \bullet S_T - H)^2\right) \\
&\quad = E\left((v_0 - V_0)^2 L_0 + L \bullet \left(\langle V, V\rangle^{P^\star} - \xi \bullet \langle V, S\rangle^{P^\star}\right)_T\right) \\
&\quad = E\left((v_0 - V_0)^2 L_0 + L \bullet \langle V - \xi \bullet S, V - \xi \bullet S\rangle^{P^\star}_T\right).
\end{aligned}
$$

Conditions for $P = P^\star$

Proposition

The following statements are equivalent:

1. $P^\star = P$
2. K *(or* L*) is a predictable process of finite variation and* L_0 *is deterministic.*
3. $K = \hat{K}$ *and* L_0 *is deterministic.*
4. $\mathscr{E}(\hat{K})_T$ *is finite and deterministic.*

In each case $L := \mathscr{E}(\hat{K})/\mathscr{E}(\hat{K})_T$ *and* $Q^\star = Q$

This extends Schweizer (1996) to discontinuous $\hat{K}$

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